Symmetrical Pentomino Pairs

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Pentominoes – shapes made of five congruent squares – provide a natural platform for games and puzzles. In this article, I describe my own experiences designing pentomino packing games, and some interesting puzzles that emerged when studying symmetrical pairs of pentominoes.

1 Introduction

THE twelve *pentominoes* (Figure 1) were originally defined, named and studied by Solomon W. Golomb in his seminal book, *Polyominoes* [1]. They have been copiously studied, played with, explored and documented as a popular form of recreational mathematics, by thousands of enthusiasts.

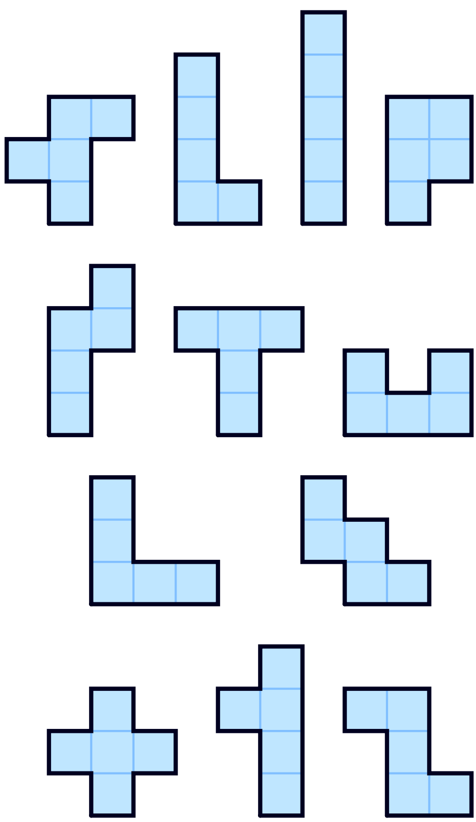

Figure 1. The pentominoes: FLIPNTUVWXYX.

The twelve pentominoes consist of all the distinct ways in which five congruent squares can be joined evenly on their edges. Golomb named them for letters of the alphabet they most resemble: FLIPNTUVWXYZ (some are best viewed diagonally). For ease of remembering, they spell 'FLIP' 'N' a straight flush from T to Z.

One of the many interesting aspects of pentominoes is that certain pairs of them can fit together to form symmetrical shapes. We have done a thorough exploration of all their possible combinations and report the results herewith.

2 Background

In 1976, I read a science fiction novel by Arthur C. Clarke, *Imperial Earth* [2], which featured pentominoes as an important part of the plot. Clarke had been bitten by the pentomino bug from an article by Martin Gardner in the November 1960 issue of *Scientific American* [3].

Clarke's addiction quickly passed on to me, and in 1979 we formed Kadon Enterprises, Inc. [4], to produce and sell a pentominoes set named Quintillions® (Figure 2). This consisted of twelve laser cut wood blocks, each equivalent to five congruent cubes in all their possible different planar adjacencies, hence *planar pentacubes*.

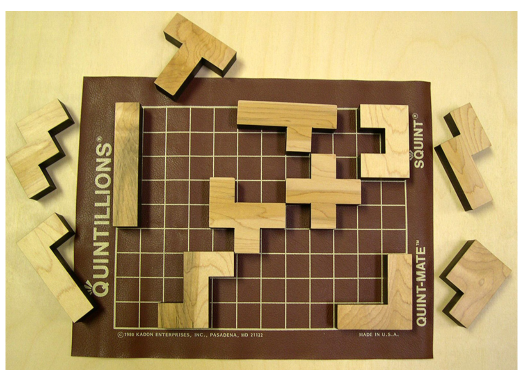

Figure 2. The Quintillions® set.

The laser cutting technique was chosen to assure the best precision of size and accurate fit even in three-dimensional assemblies and with minimum wasted material. For strength and aesthetics, we needed to have the wood grain follow the longest dimension of each piece. Since none of the known rectangles – 4×15, 3×20, 5×12 and 6×10 – yielded solutions with all pieces placed lengthwise, we worked out the next best thing, a 4×16 solution, by hand, with four cubes left over.

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Exploring various solutions at one point suggested that pairs of pieces often formed a symmetrical shape. The question of how many combinations were possible became the subject of this research project in 1981.

3 Exhaustive Search

A systematic approach required testing each pentomino connected to every other pentomino, thus $\binom{12}{2} = 66$ combinations, and for each such pair we tested all ways of joining at least two of their edges. It was interesting to observe that some pairs could combine into more than one symmetrical shape, and that some symmetrical shapes could be formed by more than one pair.

Here, then, are all 52 symmetrical pentomino pairs. The first group (Figure 3) shows the orthogonal twins, in which more than one pair can make the same shape.

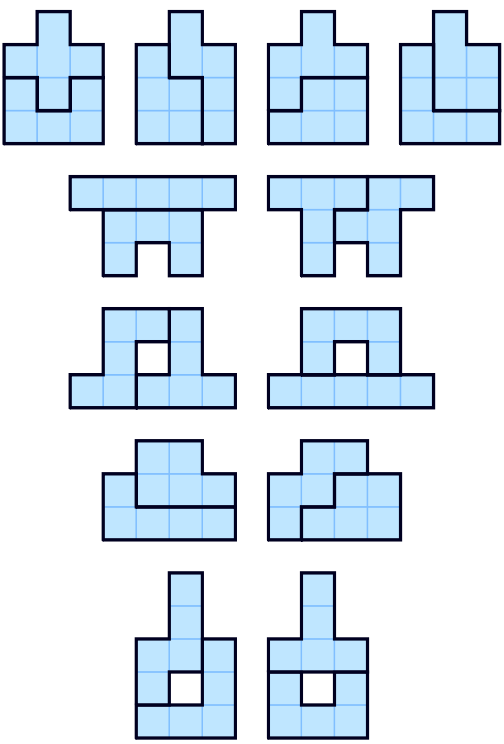

Figure 3. The twelve orthogonal twins.

The second group (Figure 4) contains the diagonally symmetrical twins. These were more difficult to identify, especially those that have holes. The most elusive was the YP pair, which is a copy of the symmetrical XW pair.

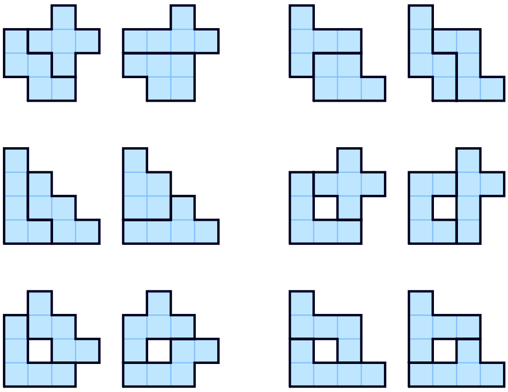

Figure 4. The twelve diagonal twins.

Towards the end of the search, we kept testing pairs made of symmetrical pieces to see whether they had twins among the asymmetrical pieces. This turned up the really surprising pairs. Figure 5 shows the group of orthogonal symmetrical pairs with no twins.

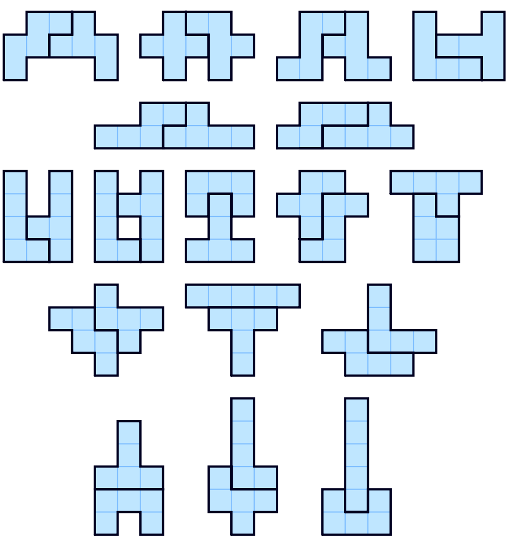

Figure 5. Orthogonal solo pairs.

The last group, shown in Figure 6, has the most ornate diagonal symmetries. Six of the pairs {VW, VX, LZ, NV, FP, NP} can form more than one shape.

The 52 different symmetrical pentomino pairs are composed of 34 essential pairings. Several pairs can make two shapes, and NV and YP can make three. If we disregard the requirement that at least two unit edges touch, there are ten additional pairs. Working them out is left as an exercise (*hint:* they use the orthogonally symmetrical I, T, X and U).

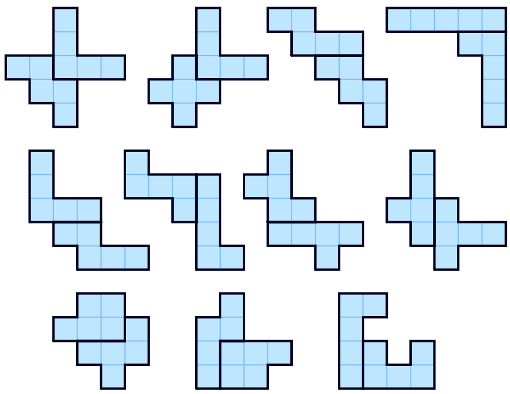

Figure 6. The seventeen diagonal solo pairs.

4 Simultaneous Pairs

Once we identified all the possible symmetrical pairs, we began to wonder if we could make six pairs simultaneously, and how many ways that could be done.

Not having the luxury of a computer, we started to sort them manually, beginning with the I pentomino, which has the fewest mates, based on this 'congeniality' index of how many other pentominoes any particular pentomino can have as a partner:

I $\rightarrow$ {L, T, U}

U $\rightarrow$ {I, L, T, X}

X $\rightarrow$ {F, U, V, W}

Z $\rightarrow$ {F, L, N, T}

N $\rightarrow$ {N, P, T, W, X}

Y $\rightarrow$ {L, N, P, T, U}

L $\rightarrow$ {F, I, U, W, Y, Z}

N $\rightarrow$ {F, P, V, W, Y, Z}

W $\rightarrow$ {F, L, N, P, V, X}

F $\rightarrow$ {N, P, T, W, X, Y, Z}

P $\rightarrow$ {F, L, N, T, V, W, Y}

T $\rightarrow$ {F, I, P, U, V, Y, Z}

After two days of struggling with paper and pencil notes, it occurred to us that there had to be an easier way. Our friend Jim Kolb was a BASIC programmer, and we appealed to his expertise to speed up our search.[1] We provided him with the 52 pairings, and in short order his program turned up some interesting statistics:

- The total number of six simultaneous symmetrical pairs is a prime number: 2,153.
- Of all these, there is exactly one combination in which all six pairs can form more than one shape.
- There are 8 solutions in which all six pairs form only one shape.
- There are 2 solutions in which all six pairs are orthogonal.
- Even though there are so many diagonal pairs to play with, it is not possible to form six of them simultaneously.

Having conquered the question of six simultaneous symmetrical pairs, we took the next step of making larger symmetrical figures by combining all the pairs, such as the 'Thunderbird' (Figure 7) and 'Christmas Tree' (Figure 8). Solving the 'Christmas Tree' is left as an exercise.

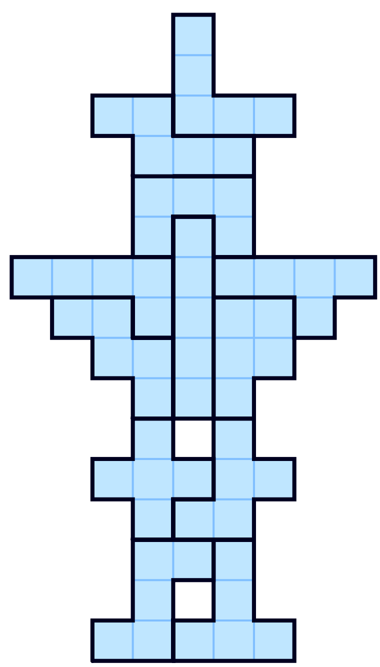

Figure 7. 'Thunderbird' pattern made of six symmetrical pairs of pentominoes.

[1]Personal correspondence, 9 September 1981. The nine-page list of 2,153 sets of six simultaneous symmetrical pairs took 10 hours to calculate on an ATARI 800.

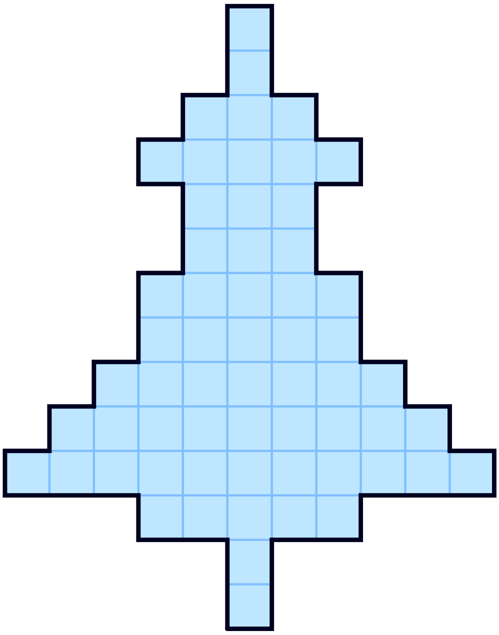

Figure 8. 'Christmas Tree' solvable with six symmetrical pairs of pentominoes.

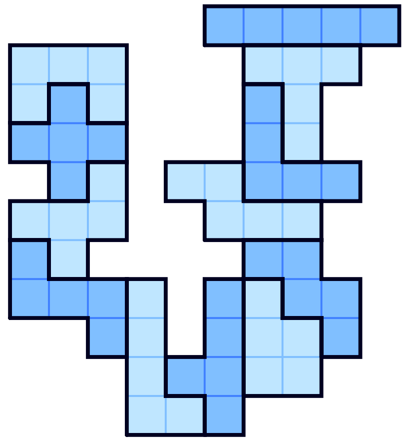

Figure 9. A chain of eleven symmetrical pairs with perimeter 72.

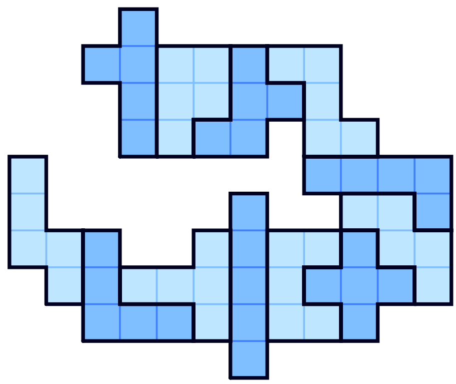

Figure 10. A chain of eleven symmetrical pairs with perimeter 68.

5 An Unsolved Problem

When exploring any game or puzzle idea, the human brain is never content to leave well enough alone, always pushing for the next level or a new variation. We are born with such inquisitiveness and a propensity for design and analysis. The 'symmetrical pairs' problem is no exception.

Having established the inventory of pairs and even played with forming combined constructions, we wondered:

> *Is it possible to arrange the pentominoes such that each is part of a symmetrical pair with every piece that it touches?*

Pushing the boundaries even further, we can ask whether a chain of twelve symmetrical pairs is possible, such that the two end pieces actually meet and form the last symmetrical pair. This question has not yet been answered. Perhaps an enterprising reader will find such a solution or prove that none exists before our own research finds the answer.

Figures 9 and 10 illustrate such chains of symmetrical pairs. Furthermore, both of them, like dominoes, have end pieces that are also compatible, although with intervening spaces. It would as well be interesting to explore the shortest and longest perimeters of such constructions.

6 A Pairs-Rich Rectangle

Once we had identified all symmetrical pairs of pentominoes, another idea raised its head:

> *What is the maximum number of symmetrical pairs in a* 5×12 *rectangle?*

A day and a half of solving by hand yielded the following composition (Figure 11), which became the Quintillions® box pattern:

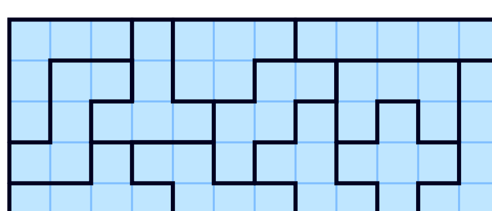

Figure 11. A symmetries-packed rectangle.

Not only does this rectangle contain five symmetrical pairs (Figure 12) but three symmetrical triples (Figure 13), and Kadon's logo [4] – the W pentomino – occupies the centre. The U is part of three overlapping pairs and two entwined triples.

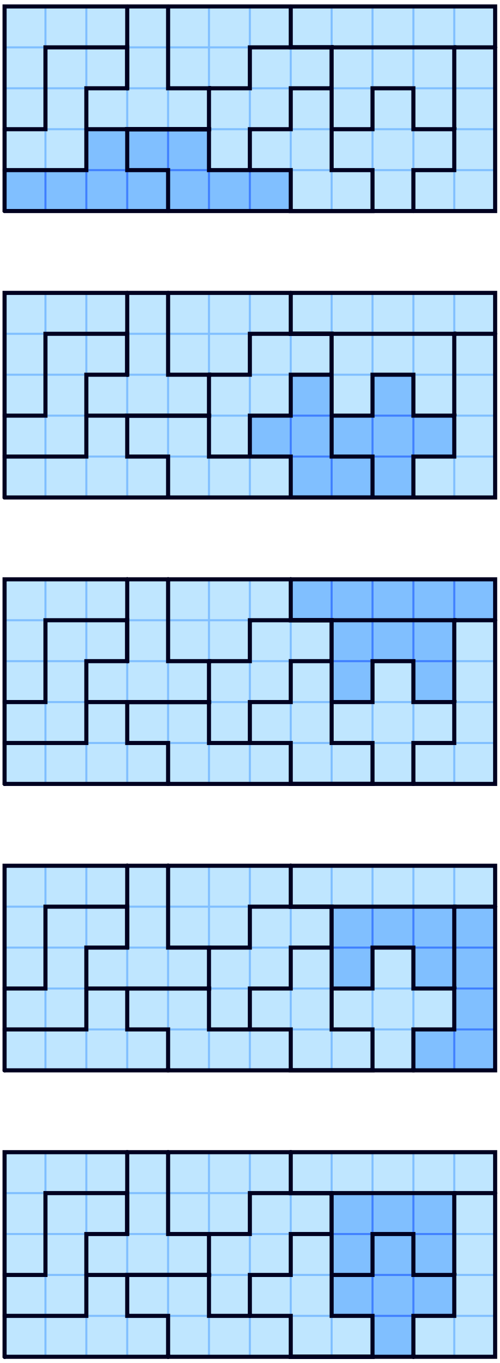

Figure 12. Five symmetrical pairs embedded in the 5×12 packing (shaded).

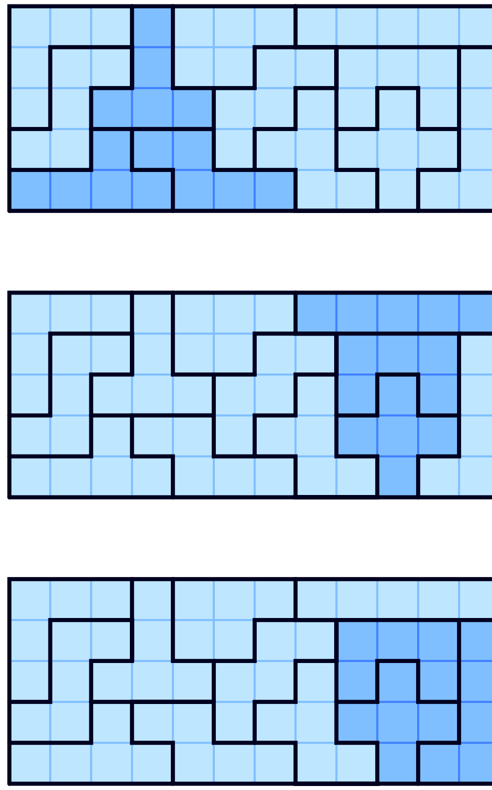

Figure 13. Three embedded symmetrical triples embedded in the 5×12 packing (shaded).

7 Conclusion

Pentominoes are one of the most enduring and fascinating members of the recreational math genre. They present a virtually endless source of activities and explorations. The project reported here explored one particular aspect of pentominoes: how the twelve distinct pentominoes can form symmetrical pairs.

In designing this research, we exhaustively examined the relationships between any two pentominoes (66 distinct pairs, in all possible joinings) to catalog every possibility. No computer program was used in their derivation, but a further investigation into forming six simultaneous symmetrical pairs did use a search program to identify the 2,153 ways that six pairs can be formed.

Because some pentominoes can pair with only a few others, placing them first leaves the more versatile or 'congenial' pieces for last. This gives the solver a better chance of success when looking for solutions that require six symmetrical pairs.

Why symmetry? There is something intuitively appealing and aesthetically pleasing in symmetry; a balance, a rhythm. Tessellations and recurring patterns provide a reassuring visual environment, and creating them is intellectually rewarding. The result is elegance, beauty, and open ended engagement with each discovery.

The 'symmetrical pentomino pairs' project was an absolute delight, and it leaves one unsolved problem still on the table: the closed chain of twelve simultaneous symmetrical pairs. We look forward to hearing from readers with a good answer.

References

[1] Golomb, W. S., *Polyominoes*, New York, Scribners, 1965.

[2] Clarke, A. C., *Imperial Earth*, New York, Ballantine Books, 1976.

[3] Gardner, M., 'Mathematical Games: More About the Shapes that Can Be Made with Complex Dominoes', *Scientific American*, no. 203, November 1960, pp. 186–198.

[4] Jones, K. *Kadon Enterprises, Inc.*, 1979. http://www.gamepuzzles.com

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Heptalion Challenge #4

Use all tiles to cover all symbols. See 'Heptalion' (p. 17) for details.