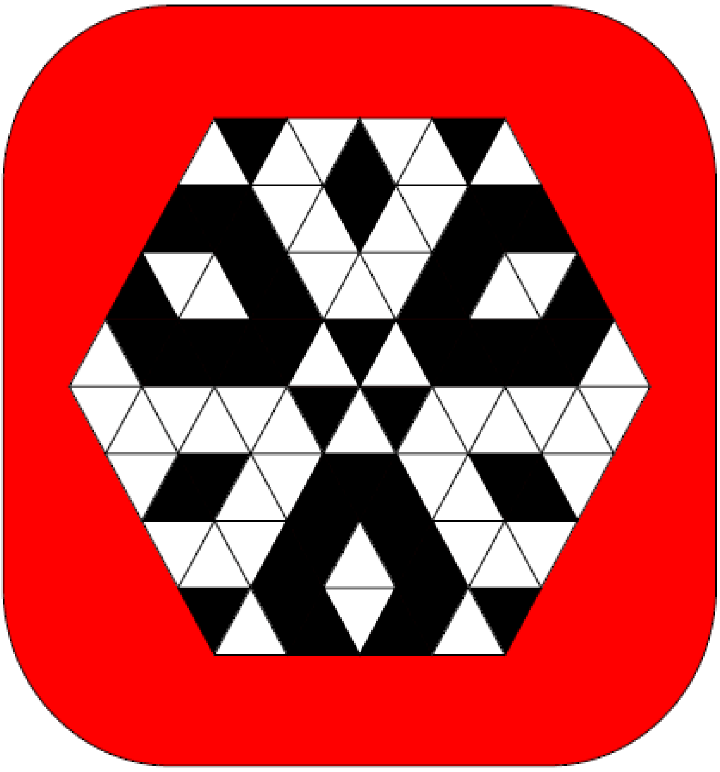
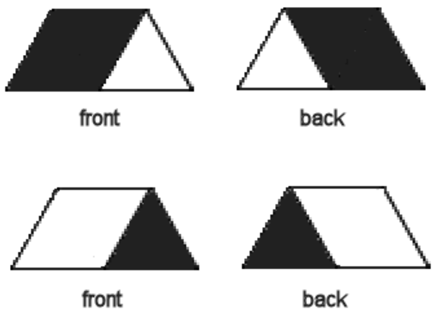
INTARSIA



Created by Henrik Morast *Developed by Kate Jones* *This solution by Robin King*

Intarsia® is a set of 32 trapezoidal, half-hexagon tiles: 16 of the tiles are made of a black diamond joined with a white triangle, and 16 are made of a white diamond joined with a black triangle. All tiles are reversible and can be flipped either side up.



The two types of Intarsia tiles, flipped both ways.
Each can be oriented in 6 different directions.

INTRODUCTION

by Henrik Morast

A combinatorial puzzle problem

The 32 tiles can be arranged to form a hexagon. The straightforward question is in how many ways this can be done, *i.e.*, in how many different ways is it possible to arrange the 32 Intarsia tiles into a hexagon?

The approach taken to find the solution was to divide the problem into two parts:

1. In how many ways can a pattern of 32 equally sized half-hexagon trapezoids be arranged to create a hexagon *(the Pattern problem)*?
2. In how many visually different ways can each of the patterns found be populated by a set of 32 Intarsia tiles *(the Tiling problem)*?

The Pattern Problem

To find the solution of the pattern problem, we used a computer. We divided the hexagon to be filled into equilateral, numbered triangles matching the size of the triangle of an Intarsia tile. The idea was to use brute force and simply loop through each triangle and all possible ways to cover it with a trapezoid under the following conditions:

- The triangle has not been covered already;
- The covering trapezoid is placed inside the hexagon (thus also covering two other triangles); and
- The trapezoid cannot cover any part of another trapezoid.

Each possible arrangement was examined recursively until all triangles had been covered (in which case a valid pattern was found and counted) or until a triangle could not be covered subject to the three conditions above (in which case that particular arrangement was abandoned).

The program was verified by computing in how many ways two trapezoids can form a first-order hexagon (3) and in how many ways eight trapezoids can form a second-order hexagon (9). Those numbers can easily be confirmed "by hand". Note, however, that these are only different solutions when looked upon from a specific viewpoint. Symmetries of the board were not taken into account at this stage.

The program was also used to find that there are 12,597 different solutions for how many ways 18 trapezoids can form a third-order hexagon. At this point it became clear that the brute force approach would not be suitable for coping with 32 tiles for the full fourth-order hexagon. It was obvious that a slightly more clever approach was needed.

The new idea was to split the hexagon in two, count in how many ways each part can be filled with trapezoids, and multiply to get a total. But, as there are many possible ways to split the board, this must be repeated for a whole set of splits that together will catch all possible patterns for the complete fourth-order hexagon. To find such a set of splits, the 96-triangle hexagon was divided into three sections: a northern area of 40 triangles, a southern area of 40 triangles, and a braid of 16 separating triangles in the center (see figure 2).

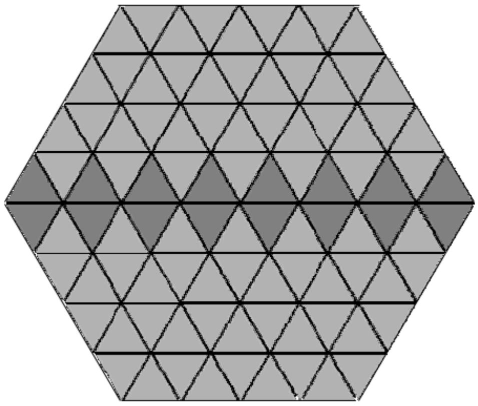


Figure 2. The hexagon divided into three sections.

The center section is truly separating, because a trapezoid covering a triangle in the northern area cannot also cover a triangle in the southern area. As a result, iterating through all possible combinations of assigning each single triangle in the center section to either the northern area or the southern area will give a complete set of ways to split the hexagon in two—complete here in the sense that the resulting set will be sufficient for catching all patterns for the complete fourth-order hexagon by the suggested method.

However, most of the resulting combinations of assigning triangles from the center to the north and south area cannot be used for creating valid patterns. As each trapezoid will cover exactly three triangles, any area covered with trapezoids must comprise a number of triangles that is a multiple of three. There are 40 northern area triangles, so only the combinations assigning 2, 5, 8, 11 or 14 triangles from the center need to be considered. More than this, half of the triangles in the center section are only meaningful to be assigned to the north if the adjacent triangle in direct contact with the northern area is also assigned to the north.

So the computer program was adapted to loop through all combinations of valid ways to assign two triangles from the center to the northern area. For each combination the number of ways 14 trapezoids could be fitted was counted. The corresponding was also performed for each complementary southern area (*i.e.*, in how many ways could 18 trapezoids be fitted). Multiplying each pair and summing them up gave a number of valid patterns for the complete hexagon, which was multiplied by two to cover the equivalent case of 14 center triangles assigned to the north and two to the south.

Thereafter, the same thing was done for the case of assigning five center triangles to the north and 11 to the south (and vice versa). Finally, the even distribution of eight triangles in both directions was attacked. For this final case, the symmetric patterns, where the southern area rotated 180 degrees is identical to the northern area, were counted separately.

Summing up all valid patterns found gave a stunning 61,781,885 non-symmetric and 17,700 symmetric patterns. But these numbers do not take the six-fold symmetry of the hexagon into account. The number of non-symmetric solutions should be divided by six, and the number of symmetric patterns should be divided by three. This gives a total of 10,296,981 different non-symmetric and 5,900 symmetric patterns.

Note: Mirror patterns have been regarded as different in this examination.

The Tiling Problem

Now, having found the number of ways a pattern of 32 equally sized half-hexagon trapezoids can be arranged to create a hexagon, the question is—in how many ways can these patterns be populated by a set of 32 Intarsia tiles? Again, there is a need to take symmetry into account. Therefore we have to examine four different cases where only the last one will result in true pattern and color symmetry.

1. **Populating non-symmetric patterns**

   The number of ways to populate a non-symmetric pattern with Intarsia tiles is equivalent to the number of ways you can choose exactly 16 out of 32 trapezoids that shall have black diamonds {= 32!/(16!*16!)} and multiply by all combinations that the 32 tiles can be flipped {=2^32}. The number of solutions for this category is:

   10,296,981 * 32!/(16!*16!) * 2^32 = 26,582,898,445,720,652,234,096,640

2. **Populating symmetric patterns with a non-symmetric distribution of tiles**

   Again there are 32!/(16!*16!) ways to choose the trapezoids populated by black diamond Intarsia tiles. However, for this subgroup of the 5,900 symmetric patterns, we do not want to count symmetric distributions. The number of symmetric distributions is given by choosing exactly 8 out of 16 trapezoids in the northern part of the hexagon to be populated by black diamond tiles {=16!/(8!*8!)}. Then, as before, each tile can be flipped in two ways. But due to symmetry aspects we must divide by two, so the total number of different solutions for this category is:

   5,900 * (32!/(16!*16!) – 16!/(8!*8!)) * 2^32 / 2 = 7,615,617,756,209,086,464,000

3. **Populating symmetric patterns with a symmetric distribution of tiles but flipping the tiles in a non-symmetric way.**

   For the symmetric distribution of tiles in symmetric patterns we now want to calculate the number of ways tiles can be flipped that will not produce a symmetric result. We need to count all combinations of flipping and reduce this number by all symmetric flipping combinations. Again due to symmetry we must divide by 2:

   5,900 * 16!/(8!*8!) * (2^32 – 2^16) / 2 = 163,062,387,671,040,000

4. **Populating symmetric patterns symmetrically (*i.e.*, with a symmetric distribution of tiles and a symmetric flipping of tiles)**

   Finally, the number of true symmetric solutions is:

   5,900 * 16!/(8!*8!) * 2^16) = 4,976,345,088,000

Now, summing up all patterns found in 1, 2, 3, and 4 provides the answer to our problem. The number of different ways 32 Intarsia tiles can be arranged into a hexagon is:

26,590,514,226,544,225,336,688,640

And among this astronomical number of possibilities, the solution shown at the top has the most exquisite multiple symmetries: orthogonal, diagonal, rotational, mirror and opposite-color symmetries along different axes. The special challenge of placing the trapezoidal tiles into the hexagon is that there are 16 of each kind of tile, and 16 is not divisible by either 6 or 3.
